Solution 4

1. Prove Hölder's Inequality in vector form: For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $p > 1$ and q conjugate to p,

$$|\mathbf{x} \cdot \mathbf{y}| \le \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p} \left(\sum_{j=1}^{n} |y_j|^q \right)^{1/q} .$$

You may prove it directly or deduce it from its integral form by choosing suitable functions f and g.

Solution. Dividing $[0, 1]$ equally into n many subintervals I_j and set $f(x) = x_j, g(x) = y_j$, for $x \in (x_j, x_{j+1}]$, Hölder's inequality for vectors follows from the same inequality for f and g.

2. Prove Minkowski's Inequality in vector form: For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $p > 1$,

$$\|\mathbf{x} + \mathbf{y}\|_p \le \|\mathbf{x}\|_p + \|\mathbf{y}\|_p .$$

You may prove it directly or deduce it from its integral form by choosing suitable functions f and g.

Solution. Same as in the previous problem.

3. Prove the generalized Hölder's Inequality: For $f_1, f_2, \cdots, f_n \in R[a, b]$,

$$\int_a^b |f_1 f_2 \cdots f_n| dx \le \left(\int_a^b |f_1|^{p_1} \right)^{1/p_1} \left(\int_a^b |f_2|^{p_2} \right)^{1/p_2} \cdots \left(\int_a^b |f_n|^{p_n} \right)^{1/p_n} ,$$

where

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1, \quad p_1, p_2, \cdots, p_n > 1 .$$

Solution. Induction on n. $n = 2$ is the original Hölder, so it holds. Let

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_{n+1}} = 1 .$$

First, using the original Hölder, we have

$$\int_a^b |f_1 f_2 \cdots f_{n+1}| \, dx \le \left(\int_a^b |f_1|^{p_1} \, dx \right)^{1/p_1} \left(\int_a^b |f_2 \cdots f_{n+1}|^q \, dx \right)^{1/q} ,$$

where q is conjugate to p_1. It is easy to see

$$1 = \frac{q}{p_2} + \cdots + \frac{q}{p_{n+1}} .$$

By induction hypothesis,

$$\int_a^b |f_2^q \cdots f_{n+1}^q| \, dx \le \left(\int_a^b |f_2|^{p_2} \, dx \right)^{1/p_2} \cdots \left(\int_a^b |f_{n+1}|^{p_{n+1}} \, dx \right)^{1/p_{n+1}} ,$$

done.

4. Show that for $1 \le p < r \le \infty$,

(a)
$$\|\mathbf{x}\|_p \leq n^{\frac{1}{p}-\frac{1}{r}}\|\mathbf{x}\|_r \ ,$$

(b)
$$\|\mathbf{x}\|_r \leq n^{\frac{1}{r}}\|\mathbf{x}\|_p.$$

Solution. (a)

$$\begin{aligned}\|\mathbf{x}\|_p^p &= \sum |x_j|^p \\ &\leq \left(\sum |x_j|^{p\frac{r}{p}}\right)^{\frac{p}{r}}\left(\sum 1^{\frac{r}{r-p}}\right)^{\frac{r-p}{r}} \\ &= n^{\frac{r-p}{r}}\|\mathbf{x}\|_r^p\end{aligned}$$

so
$$\|\mathbf{x}\|_p \leq n^{\frac{1}{p}-\frac{1}{r}}\|\mathbf{x}\|_r \ .$$

(b) First of all, $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_p$. Then,

$$\begin{aligned}\|\mathbf{x}\|_r &\leq (n\|\mathbf{x}\|_\infty^r)^{\frac{1}{r}} \\ &\leq n^{\frac{1}{r}}\|\mathbf{x}\|_\infty \\ &\leq n^{\frac{1}{r}}\|\mathbf{x}\|_p \ .\end{aligned}$$

5. Establish the inequality, for $f \in R[a,b]$, $\|f\|_p \leq C\|f\|_r$ when $1 \leq p < r$ for some constant C.

   Solution By Holder's Inequality,

$$\int_a^b |f|^p \leq \left(\int_a^b 1\,dx\right)^{1-p/r}\left(\int_a^b |f|^{p\frac{r}{p}}\,dx\right)^{p/r} \leq C^p\|f\|_r^p \ ,$$

   where
$$C = (b-a)^{\frac{1}{p}-\frac{1}{r}} \ .$$

6. Show that there is no constant C such that $\|f\|_2 \leq C\|f\|_1$ for all $f \in C[0,1]$.

   Solution Consider the sequence

$$f_n(x) = \begin{cases} -n^3x+n, & x \in [0,1/n^2], \\ 0, & x \in (1/n^2,1]. \end{cases}$$

   We have $\|f_n\|_1 = 1/(2n) \to 0$ as $n \to \infty$, but $\|f_n\|_2 = 1/\sqrt{3}$ for all n. Hence, it is impossible to have some C satisfying $\|f\|_2 \leq C\|f\|_1$ for all f.

   Note. In general, it is impossible to find a constant C such that $\|f\|_r \leq C\|f\|_p, p < r$, for all f.

7. Show that $\|\cdot\|_p$ is no longer a norm on $\mathbb{R}^n$ for $p \in (0,1)$.

   Solution Again (N3) is bad. Consider two n-tuples $\mathbf{x} = (1,0,0,\ldots,0)$ and $\mathbf{y} = (0,1,0,\ldots,0)$. We have $\|\mathbf{x}+\mathbf{y}\|_p = 2^{1/p}$ but $\|\mathbf{x}\|_p = \|\mathbf{y}\|_p = 1$, so $\|\mathbf{x}+\mathbf{y}\|_p > \|\mathbf{x}\|_p + \|\mathbf{y}\|_p$.

8. In a metric space (X, d), its metric ball is the set $\{y \in X : \ d(y, x) < r\}$ where x is the center and r the radius of the ball. May denote it by $B_r(x)$. Draw the unit metric balls centered at the origin with respect to the metrics d_2, d_∞ and d_1 on $\mathbb{R}^2$.

   Solution. The unit ball $B_1^2(0)$ is the standard one, the unit ball in d_∞-metric consists of points (x, y) either $|x|$ or $|y|$ is equal to 1 and $|x|, |y| \leq 1$, so $B_1^\infty(0)$ is the square of side length 2 centered at the origin. The unit ball $B_1^1(0)$ consists of points (x, y) satisfying $|x|+|y| \leq 1$, so the boundary is described by the curves $x+y = 1, x, y \geq 0,\ x-y = 1, x \geq 0, y \leq 0, -x+y = 1, x \leq 0, y \geq 0$, and $-x-y = 1, x, y \leq 0$. The result is the tilted square with vertices at $(1, 0), (0, 1), (-1, 0)$ and $(0, -1)$.

9. Determine the metric ball of radius r in (X, d) where d is the discrete metric, that is, $d(x, y) = 1$ if $x \neq y$.

   Solution. When $r \in (0, 1]$, $B_r(x) = \{x\}$. When $r > 1, B_r(x) = X$.

10. Consider the functional Φ defined on $C[a, b]$

$$\Phi(f) = \int_a^b \sqrt{1 + f^2(x)}\, dx.$$

   Show that it is continuous in $C[a, b]$ under both the supnorm and the L^1-norm. A real-valued function defined on a space of functions is traditionally called a functional.

   Solution. Let $h(y) = \sqrt{1+y^2}$. Then $\Phi(f) = \int_a^b h(f)dx$. Since $h'(y) = \dfrac{y}{\sqrt{1+y^2}} \leq 1$, one has, by the mean value theorem

$$\begin{aligned}|\Phi(f) - \Phi(g)| &\leq \int_a^b |h(f) - h(g)|dx \leq \int_a^b |f - g| \max_{s\in(g,f)} |h'(s)|dx \\ &\leq \int_a^b |f - g|dx.\end{aligned}$$

   Hence it is continuous in $C[a, b]$ under the d_1-distance. As d_∞ is stronger than d_1, the functional is also continuous in d_∞.

11. Consider the functional Ψ defined on $C[a, b]$ given by $\Psi(f) = f(x_0)$ where $x_0 \in [a, b]$ is fixed. Show that it is continuous in the supnorm but not in the L^1-norm. Suggestion: Produce a sequence $\{f_n\}$ with $\|f_n\|_1 \to 0$ but $f_n(x_0) = 1$, $\forall n$. Ψ is called an evaluation map.

   Solution. Take $[a, b] = [-1, 1]$ and $x_0 = 0$. Note $|\Psi(f) - \Psi(g)| = |f(0) - g(0)| \leq \max_{x\in[-1,1]} |f(x) - g(x)|$. Hence it is continuous in the d_∞-metric. Let f_n be a continuous function such that $f_n(x) = 1, x \in [-1/n, 1/n]$; $f_n(x) = 0, x \in [-2/n, 2/n]$, and $0 \leq f_n \leq 1$. Then $\Psi(f_n) = 1$ but $f_n \to 0$ in the d_1-metric.